The Niels Henrik Abel Contest 1997–98

FINAL

March 12th 1998

Problem 1

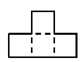
Let $a_0, a_1, a_2, \dots$ be an infinite sequence of positive integers with $a_0 = 1$ and so that $a_i^2 > a_{i-1}a_{i+1}$ for $i > 0$.

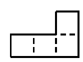
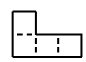
a) Prove that $a_i < a_1^i$ for all $i > 1$.

b) Prove that $a_i > i$ for all i.

Problem 2

Given a board containing $n \times n$ squares, n apositive integer. We wish to cover this board using particular shapes put together from four squares. The pieces should not overlap nor transcend the board, however, they may otherwise be placed and rotaed freely.

a) For which n may the board be covered by  shapes?

b) For which n may the board be covered if both  and  may be used?

Problem 3

Let n be a positive integer.

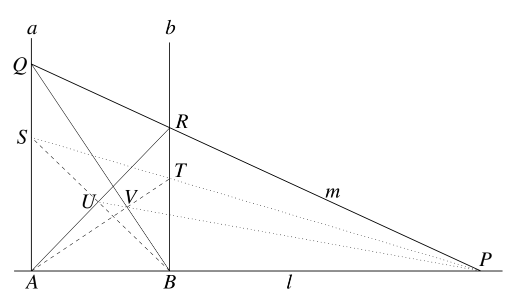
a) Prove that n divides $1^5 + 3^5 + 5^5 + \cdots + (2n-1)^5$.

b) Prove that n^2 divides $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3$.

Problem 4

Let l be a line, and let A, B, and P be points on l so that P lies outside the segment AB. Let a and b be the lines perpendicular to l and passing through A and B respectively. Draw a line m through P which is neigher parallell to or perpendicular to l; m intersects a at Q and b at R.

Let S be the point on a so that AR and BS are perpendicular and denote the intersection U. Similarly, let T be the point on b so that BQ and AT are perpendicular and denote the intersection V.



a) Prove that P, S, and T lie on a line.

b) Prove that P, U, and V lie on a line.