The Niels Henrik Abel mathematics competition: Final 2017–2018

6 March 2018 (English)

In the final round of the Abel contest there are four problems (six subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

For an odd number n, we write $n!! = n \cdot (n-2) \cdots 3 \cdot 1$. How many different residues modulo 1000 do you get from $n!!$ for $n = 1, 3, 5, \ldots$?

Problem 2

The circumcentre of a triangle ABC is called O. The points A', B', and C' are the reflections of O in BC, CA, and AB, respectively. Show that the three lines AA', BB', and CC' meet in a common point.

Problem 3

a. Find all polynomials P such that $P(x) + 3P(x+2) = 3P(x+1) + P(x+3)$ for all real numbers x.

b. Find all polynomials P such that

$$P(x) + \binom{2018}{2}P(x+2) + \cdots + \binom{2018}{2016}P(x+2016) + P(x+2018)$$
$$= \binom{2018}{1}P(x+1) + \binom{2018}{3}P(x+3) + \cdots + \binom{2018}{2015}P(x+2015) + \binom{2018}{2017}P(x+2017)$$

for all real numbers x.

The binomial coefficients are given by $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Problem 4

a. A sequence $a_1, a_2, ..., a_k$ of integers is called *valid* if for $j = 1, 2, ..., k-1$ the following holds:

- if a_j is an *even* number, then $a_{j+1} = a_j/2$, but
- if a_j is an *odd* number, then $|a_{j+1} - a_j| = 1$.

Find the smallest k such that there exists a valid sequence with $a_1 = 2018$ and $a_k = 1$.

b. Find the smallest K such that for each $n \in \{1, 2, 3, \dots, 2018\}$ there exists a valid sequence with $a_1 = n$, $a_k = 1$, and $k \le K$.