English

The Niels Henrik Abel mathematics competition 2013–2014

Final 4 March 2014

In the final round of the Abel contest there are four problems (six subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

a. Assume that $x, y \geq 0$. Show that

$$x^2 + y^2 + 1 \leq \sqrt{(x^3 + y + 1)(y^3 + x + 1)}.$$

b. Find all functions $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ which satisfy

$$(1 + y)f(x) - (1 + x)f(y) = yf(x/y) - xf(y/x)$$

for all real $x, y \neq 0$, and which take the values $f(1) = 32$ and $f(-1) = -4$.

Problem 2

The points P and Q lie on the sides BC and CD of the parallelogram $ABCD$ so that $BP = QD$. Show that the intersection point between the lines BQ and DP lies on the line bisecting $\angle BAD$.

Problem 3

a. A grasshopper is jumping about in a grid. From the point with coordinates (a, b) it can jump to either $(a + 1, b)$, $(a + 2, b)$, $(a + 1, b + 1)$, $(a, b + 2)$ or $(a, b + 1)$. The grasshopper starts in $(0, 0)$. In how many ways can it reach the line $x + y = 2014$?

b. Nine points are placed on a circle. Show that it is possible to colour the 36 chords connecting them using four colours so that for any set of four points, each of the four colours is used for at least one of the six chords connecting the given points.

Problem 4

Find all triples (a, b, c) of positive integers for which $\dfrac{32a + 3b + 48c}{4abc}$ is also an integer.