The Niels Henrik Abel mathematics competition: Final 2020–2021

8 March 2021 (English)

In the final round of the Abel contest there are four problems (eight subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.** Your answer sheets will need to be canned to be sent to the jury, so please use writing implements providing good readability.

You can score up to 10 points for each problem. Maximum score is 40.

No aids other than writing paper, writing tools (including compass and ruler, but not a protractor) and bilingual dictionaries are permitted.

Do not write your name on the answer sheets. Instead, write it on a separate sheet that you put at the top of the submitted answers. The jury will not see this sheet.

Problem 1

a. A $3n$-*table* is a table with three rows and n columns containing all the numbers $1, 2, \ldots, 3n$. Such a table is called *tidy* if the n numbers in the *first row* appear in ascending order from left to right, and the three numbers in *each column* appear in ascending order from top to bottom. How many tidy $3n$-tables exist?

b. Pål has more chickens than he can manage to keep track of. Therefore, he keeps an index card for each chicken. He keeps the cards in ten boxes, each of which has room for 2021 cards.

Unfortunately, Pål is quite disorganized, so he may lose some of his boxes. Therefore, he makes several copies of each card and distributes them among different boxes, so that even if he can only find seven boxes, no matter *which* seven, these seven boxes taken together will contain at least one card for each of his chickens.

What is the largest number of chickens Pål can keep track of using this system?

The problems continue on the next page

Problem 2

a. Show that for all $n \geq 3$ there are n different positive integers $x_1, \ldots, x_n$ such that
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

b. If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real numbers satisfying $a_1^2 + \cdots + a_n^2 \leq 1$ and $b_1^2 + \cdots + b_n^2 \leq 1$, show that
$$\big(1 - (a_1^2 + \cdots + a_n^2)\big)\big(1 - (b_1^2 + \cdots + b_n^2)\big) \leq \big(1 - (a_1 b_1 + \cdots + a_n b_n)\big)^2$$

Problem 3

a. For which integers $0 \leq k \leq 9$ do there exist positive integers m and n so that the number $3^m + 3^n + k$ is a perfect square?

b. We say that a set S of natural numbers is *synchronous* provided that the digits of a^2 are the same (in occurence and numbers, if differently ordered) for all numbers a in S. For example, $\{13, 14, 31\}$ is synchronous, since we find $\{13^2, 14^2, 31^2\} = \{169, 196, 961\}$. But $\{119, 121\}$ is not synchronous, for even though $119^2 = 14161$ and $121^2 = 14641$ have the same digits, they occur in different numbers. Show that there exists a synchronous set containing 2021 different natural numbers.

Problem 4

a. A tetrahedron $ABCD$ satisfies $\angle BAC = \angle CAD = \angle DAB = 90^\circ$. Show that the areas of its faces satisfy the equation
$$\text{area}(BAC)^2 + \text{area}(CAD)^2 + \text{area}(DAB)^2 = \text{area}(BCD)^2.$$

b. The tangent at C to the circumcircle of triangle ABC intersects the line through A and B in a point D. Two distinct points E and F on the line through B and C satisfy
$$|BE| = |BF| = \frac{\big||CD|^2 - |BD|^2\big|}{|BC|}.$$
Show that either $|ED| = |CD|$ or $|FD| = |CD|$.